University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 4

1. (a) The quadrupole term gives rise to a potential of the form,

$$V(r,\theta) = \frac{Y_{20}(\theta)}{5\epsilon_0 r^3} \times \int d\tau' \rho(r',\theta')(r')^2 Y_{20}^*(\theta') = \frac{\frac{3}{2}\cos^2\theta - \frac{1}{2}}{4\pi\epsilon_0 r^3} \int d\tau' \frac{1}{2}\left(3(r'\cos\theta')^2 - (r')^2\right)\rho(r',\theta')$$
$$= \frac{1}{4\pi\epsilon_0 r^3}\frac{1}{2}\left[\int d\tau' \left(3(z')^2 - (r')^2\right)\rho(r',\theta')\right]\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = \frac{\frac{1}{2}Q_{zz}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}{4\pi\epsilon_0 r^3}, \tag{1}$$

where we use the fact that $r'\cos\theta' = z'$ and that,

$$Q_{zz} = \int d\tau' \left(3(z')^2 - (r')^2\right)\rho(r',\theta'). \tag{2}$$

(b) Just as above, we have,

$$V(r,\theta) = \frac{Y_{00}(\theta)}{\epsilon_0 r} \times \int d\tau' \rho(r',\theta') Y_{00}^*(\theta') + \frac{Y_{10}(\theta)}{3\epsilon_0 r^2} \times \int d\tau' \rho(r',\theta')(r') Y_{10}^*(\theta')$$
$$= \frac{1}{4\pi\epsilon_0 r}\int d\tau' \rho(r',\theta') + \frac{1}{4\pi\epsilon_0 r^2}\int d\tau' \rho(r',\theta') r'\cos\theta' = \frac{Q}{4\pi\epsilon_0 r} + \frac{p_z}{4\pi\epsilon_0 r^2}, \tag{3}$$

where we've used the fact that,

$$Q = \int d\tau' \rho(r',\theta') \tag{4}$$

$$p_z = \int d\tau' \rho(r',\theta') z' = \int d\tau' \rho(r',\theta') r'\cos\theta'. \tag{5}$$

2. (a) We consider the force on a pointlike, permanent dipole due to the presence of an externally applied electric field $\vec{E}$. We would like to derive this by using the fact that $\vec{F} = -\nabla U$, where $U = -\vec{p}\cdot\vec{E}$. That is, the force on a pointlike dipole due to an external field can be understood as arising from the variation of the energy of the dipole as we move it around - the dipole experiences a force in the direction in which its energy of interaction decreases the fastest. In particular, since we assume that the dipole moment is fixed as we do this, we do not differentiate the dipole vector when we use this formula. Using the product rule,

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}, \tag{6}$$

with $\vec{A} = \vec{E}$ and $\vec{B} = \vec{p}$ and the fact that $\nabla\times E = 0$ in electrostatics, we see that,

$$\vec{F} = -\nabla U = -\nabla(-\vec{p}\cdot\vec{E}) = \vec{p}\times(\nabla\times\vec{E}) + \vec{E}\times(\nabla\times\vec{p}) + (\vec{p}\cdot\nabla)\vec{E} + (\vec{E}\cdot\nabla)\vec{p} = (\vec{p}\cdot\nabla)\vec{E}. \tag{7}$$

(b) Now, using the same identity, we see that an extra term survives as from Ampere's Law it is clear that $\vec{B}$ does not have vanishing curl,

$$\vec{F} = -\nabla U = -\nabla(-\vec{m}\cdot\vec{B}) = \vec{m}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{m}) + (\vec{m}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{m} = \vec{m}\times(\nabla\times\vec{B}) + (\vec{m}\cdot\nabla)\vec{B}. \tag{8}$$

3. (a) We will get to the general expression by using the expression we derived in class and changing coordinates in the following way. First, we note that since $p\cos\theta = \vec{p}\cdot\hat{\mathbf{r}}$ and $\vec{p} = p\hat{\mathbf{z}}$, we can write,

$$4\pi\epsilon_z \vec{E}(\vec{r}) = p\frac{3\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}}{r^3} = \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\vec{p}) - \vec{p}}{r^3} \tag{9}$$

Now, since $\vec{p}$, $\hat{\mathbf{r}}$, $\vec{r}$ and $\vec{E}$ are all vectors, they transform as $\vec{p}'' = R\vec{p}$, $\hat{\mathbf{r}}'' = R\hat{\mathbf{r}}$, $\vec{r}'' = R\vec{r}$ and $\vec{E}'' = R\vec{E}$ under a rotation of coordinates. Further, as dot products are invariant under rotations,

$$\hat{\mathbf{r}}''\cdot\vec{p}'' = (R\hat{\mathbf{r}})^T R\vec{p} = \hat{\mathbf{r}}^T R^T R\vec{p} = \hat{\mathbf{r}}^T\vec{p} = \hat{\mathbf{r}}\cdot\vec{p}, \tag{10}$$

and $(r')^2 = \vec{r}'' \cdot \vec{r}'' = \vec{r} \cdot \vec{r} = r^2$. Applying a rotation matrix R which takes the polarization vector $\vec{p} = p\hat{\mathbf{z}}$ to one that points in some desired direction to both sides of equation (9), we see that

$$4\pi\epsilon_z \vec{E}''(\vec{r}'') = \frac{3\hat{\mathbf{r}}''(\hat{\mathbf{r}} \cdot \vec{p}) - \vec{p}''}{r^3} = \frac{3\hat{\mathbf{r}}''(\hat{\mathbf{r}}'' \cdot \vec{p}'') - \vec{p}''}{(r'')^3}. \tag{11}$$

Now, just relabelling the primed quantities as unprimed quantities, we see that for a dipole at the origin now with with arbitrary $\vec{p}$, we must have,

$$4\pi\epsilon_z \vec{E}(\vec{r}) = \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \vec{p}) - \vec{p}}{r^3}. \tag{12}$$

Further, using the fact that the dipole moment is unchanged by translation of the origin (proved in Griffiths 3.4.3), we can translate the origin of the coordinates by $\vec{r}'' = \vec{r} + \vec{r}'$ such that the dipole is centered at $\vec{r}'$, and the above equation becomes,

$$4\pi\epsilon_z \vec{E}(\vec{r}'') = \frac{3(\hat{\mathbf{r}}'' - \hat{\mathbf{r}}')((\hat{\mathbf{r}}'' - \hat{\mathbf{r}}') \cdot \vec{p}) - \vec{p}}{|\vec{r}'' - \vec{r}'|^3}. \tag{13}$$

Again, we can relabel the double primed quantities as unprimed, and using $\hat{\mathbf{n}} = \hat{\mathbf{r}} - \hat{\mathbf{r}}'$ we find,

$$4\pi\epsilon_z \vec{E}(\vec{r}) = \frac{3(\hat{\mathbf{r}} - \hat{\mathbf{r}}')((\hat{\mathbf{r}} - \hat{\mathbf{r}}') \cdot \vec{p}) - \vec{p}}{|\vec{r} - \vec{r}'|^3} = \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{p}) - \vec{p}}{|\vec{r} - \vec{r}'|^3}. \tag{14}$$

(b) We find the energy of the configuration by evaluating the energy of one dipole in the electric field due to the other one,

$$4\pi\epsilon_0 U_{12} = -4\pi\epsilon_z \vec{p}_1 \cdot \vec{E}_2(\vec{r}_2) = -\vec{p}_1 \cdot \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} = -\frac{3(\hat{\mathbf{n}} \cdot \vec{p}_1)(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} \tag{15}$$

(c) If two dipoles are parallel to each other and their line of separation, then we have that $\hat{\mathbf{n}} \cdot \vec{p}_1 = p_1$, $\hat{\mathbf{n}} \cdot \vec{p}_2 = p_2$ and $\vec{p}_1 \cdot \vec{p}_2 = p_1 p_2$, so we have,

$$4\pi\epsilon_0 U_{12} = -\frac{3(\hat{\mathbf{n}} \cdot \vec{p}_1)(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} = -\frac{2p_1 p_2}{|\vec{r}_1 - \vec{r}_2|^3}. \tag{16}$$

As the energy of this configuration decreases if we bring the two dipoles together, we see that they attract. Now, suppose that they are parallel to each other and perpendicular to their line of separation, so $\hat{\mathbf{n}} \cdot \vec{p}_1 = 0$, $\hat{\mathbf{n}} \cdot \vec{p}_2 = 0$ and $\vec{p}_1 \cdot \vec{p}_2 = p_1 p_2$, then we have,

$$4\pi\epsilon_0 U_{12} = -\frac{3(\hat{\mathbf{n}} \cdot \vec{p}_1)(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} = \frac{p_1 p_2}{|\vec{r}_1 - \vec{r}_2|^3}, \tag{17}$$

and we see that their electrostatic interaction energy increases if we bring them closer together, indicating that they repel.

(d) Now, suppose that both dipoles are at fixed positions and that the first dipole has a fixed orientation parallel to their line of separation. Then we have that $\vec{p}_1 = \hat{\mathbf{n}} p_1$, $\hat{\mathbf{n}} \cdot \vec{p}_1 = p_1$ and $\vec{p}_1 \cdot \vec{p}_2 = p_1 \hat{\mathbf{n}} \cdot \vec{p}_2$ and

$$4\pi\epsilon_0 U_{12} = -\frac{3(\hat{\mathbf{n}} \cdot \vec{p}_1)(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} = -\frac{3p_1(\hat{\mathbf{n}} \cdot \vec{p}_2) - p_1(\hat{\mathbf{n}} \cdot \vec{p}_2)}{|\vec{r}_1 - \vec{r}_2|^3} = -\frac{2p_1(\hat{\mathbf{n}} \cdot \vec{p}_2)}{|\vec{r}_1 - \vec{r}_2|^3}. \tag{18}$$

Clearly, for fixed distance, the energy is minimized if $\vec{p}_2$ is aligned with $\hat{\mathbf{n}}$, so $\hat{\mathbf{n}} \cdot \vec{p}_2 = p_2$, and we expect that the dipoles will want to align. If we suppose instead that the the first dipole has a fixed orientation perpendicular to their line of separation, we have that $\hat{\mathbf{n}} \cdot \vec{p}_1 = 0$ and

$$4\pi\epsilon_0 U_{12} = -\frac{3(\hat{\mathbf{n}} \cdot \vec{p}_1)(\hat{\mathbf{n}} \cdot \vec{p}_2) - \vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3}. \tag{19}$$

Here, for fixed distance, the energy is minimized if $\vec{p}_2$ is anti-aligned with $\vec{p}_2$, so $\vec{p}_1 \cdot \vec{p}_2 = -p_1 p_2$.

4. (a) The bound and surface charge densities are given by,

$$\sigma_b(s=a) = \vec{P}(s=a)\cdot\hat{\mathbf{n}}(s=a) = (ka)\hat{\mathbf{s}}\cdot\hat{\mathbf{s}} = ka \tag{20}$$

$$\rho_b(s<a) = -\nabla\cdot\vec{P}(s<a) = -\nabla\cdot(ks\hat{\mathbf{s}}) = -\frac{1}{s}\frac{\partial}{\partial s}(s(ks)) = -2k \tag{21}$$

(b) As we can calculate the fields produced by this polarization just by computing the field due to the bound charges, we can use Gauss's Law for a cylindrical surface inside and outside to do the computation,

$$2\pi slE_s(s<a) = \frac{1}{\epsilon_0}\int\rho_b d\tau' = \frac{1}{\epsilon_0}2\pi l\int_0^s(-2k)s'ds' = -\frac{2\pi lks^2}{\epsilon_0} \tag{22}$$

$$2\pi slE_s(s>a) = \frac{1}{\epsilon_0}\left(\int\rho_b d\tau' + \int\sigma_b da'\right) = \frac{1}{\epsilon_0}\left(2\pi l\int_0^a(-2k)s'ds' + 2\pi la(ka)\right) = 0 \tag{23}$$

So we have,

$$E_s(s<a) = -\frac{ks}{\epsilon_0} \tag{24}$$

$$E_s(s>a) = 0 \tag{25}$$

5. (a) Outside the conducting sphere, Gauss's Law tells us that,

$$\int\vec{D}\cdot da = \int\rho_f d\tau' = Q \Rightarrow D_r(r) = \frac{Q}{4\pi r^2} \tag{26}$$

(b) Since we are in a linear dielectric material, we have a simple relation between $\vec{E}$, $\vec{D}$, and $\vec{P}$,

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E} \Rightarrow \vec{P} = (\epsilon-\epsilon_0)\vec{D}/\epsilon = (1-\frac{1}{\epsilon_r})\frac{Q\hat{\mathbf{r}}}{4\pi r^2}. \tag{27}$$

The bulk bound charge density is then,

$$\rho_b = -\nabla\cdot\vec{P} = -(1-\frac{1}{\epsilon_r})\frac{Q}{4\pi}\nabla\cdot(\frac{\hat{\mathbf{r}}}{r^2}) = -Q\frac{\epsilon_r-1}{\epsilon_r}\delta^3(\vec{r}), \tag{28}$$

which vanishes for $r>0$.

(c) Since we have a linear dielectric, we know that

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{Q\hat{\mathbf{r}}}{4\pi\epsilon r^2}. \tag{29}$$

(d) Now, using Gauss's Law for both the bound charge computed in part b) and the free charge Q, we see that,

$$\epsilon_0\int\vec{E}\cdot da = 4\pi\epsilon_0 r^2 E_r(r) = \int(\rho_f+\rho_b)d\tau' = Q - \int Q\frac{\epsilon_r-1}{\epsilon_r}\delta^3(\vec{r})d\tau' = \frac{Q}{\epsilon_r} \Rightarrow E_r(r) = \frac{Q}{4\pi\epsilon r^2}. \tag{30}$$

This result agrees with what we found in part (b), thanks to our careful treatment of the bound charge density at the origin. Now, we can understand the significance of the pointlike bound charge at the origin - it is the bound charge which appears to screen the bare charge of the conductor.

6. (a) We claim that the electric potential (and therefore the electric field) between the two spherical thin shells is exactly the same as it would be with no dielectric present. To see why this is true, first note that there is no free charge present between the two spherical shells. Now, from Griffiths Equation (4.39), the bound charge also vanishes there for a linear dielectric material,

$$\rho_b = -\nabla\cdot\vec{P} = -\nabla\cdot(\epsilon_0\frac{\chi_e}{\epsilon}\vec{D}) = -\frac{\chi_e}{1+\chi_e}\rho_f = 0. \tag{31}$$

Thus, by Gauss's Law, between the two shells we have,

$$\epsilon_0 \nabla \cdot \vec{E} = -\epsilon_0 \nabla^2 V = \rho_b + \rho_f = 0, \tag{32}$$

which means that V obeys Laplace's equation between the two shells with the boundary conditions that,

$$V(R_1) = V_1 \tag{33}$$
$$V(R_2) = V_2. \tag{34}$$

But by the first uniqueness theorem for Laplace's equation in section 3.1.5 of Griffiths, the solution to Laplace's equation in a region with boundary values for V specified is unique! [1] Thus, if we can find a solution with these boundary conditions in the absence of dielectrics, it must still be a solution in their presence. This is not hard to do by just using the spherical symmetry of this problem via Gauss's Law,

$$\epsilon_0 \int \vec{E} \cdot da = -4\pi\epsilon_0 r^2 \frac{\partial V}{\partial r} = Q \Rightarrow V(r) = \frac{Q}{4\pi\epsilon_0 r} + C \tag{35}$$

where Q and C are determined using the boundary conditions (note $V_0 = V_1 - V_2$),

$$V(R_1) = \frac{Q}{4\pi\epsilon_0 R_1} + C = V_1 \tag{36}$$
$$V(R_2) = \frac{Q}{4\pi\epsilon_0 R_2} + C = V_2 \Rightarrow Q = 4\pi\epsilon_0 V_0 \frac{R_1 R_2}{R_2 - R_1}. \tag{37}$$

Thus, the electric field is just,

$$E_r(r) = -\frac{\partial V}{\partial r} = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{R_1 R_2 V_0}{(R_2 - R_1) r^2}. \tag{38}$$

(b) We can compute the bound surface charge density by first computing the polarization. Since this is a linear dielectric, we know that the polarization is just,

$$\vec{P}(\theta < \pi/2) = (\epsilon - \epsilon_0)\vec{E} = \frac{(\epsilon_r - 1) R_1 R_2 V_0 \epsilon_0 \hat{\mathbf{r}}}{(R_2 - R_1) r^2}, \tag{39}$$

for the bottom half hemisphere filled with dielectric and zero for the top vacuum half. The bound surface charge density in dielectric filled part is,

$$\sigma_2^b(\theta < \pi/2) = \hat{\mathbf{n}} \cdot \vec{P}(\theta < \pi/2, r = R_2) = \hat{\mathbf{r}} \cdot \vec{P}(\theta < \pi/2, r = R_2) = \frac{(\epsilon_r - 1) R_1 V_0 \epsilon_0}{(R_2 - R_1) R_2}, \tag{40}$$

while the bound surface charge density in the vacuum section vanishes.

(c) To calculate the free surface charge density, we recall that since we are in a linear dielectric medium, the boundary condition for the electric field at R_1 can be written entirely in terms of the free surface charge density (Griffiths Equation (4.40)),

$$\sigma_1^f(\theta < \pi/2) = \epsilon E_r(R_1, \theta < \pi/2) = \frac{\epsilon R_2 V_0}{(R_2 - R_1) R_1} \tag{41}$$
$$\sigma_1^f(\theta > \pi/2) = \epsilon_0 E_r(R_1, \theta > \pi/2) = \frac{\epsilon_0 R_2 V_0}{(R_2 - R_1) R_1}. \tag{42}$$

Thus, we see that the free surface charge density is different on the two hemispheres.

[1] The proof there relies on the fact that if there were more than one solution with the above boundary conditions, then their difference would be a solution with V vanishing on the boundaries. But since a solution to Laplace's equation reaches its maximum and minimum values on the boundary, which are then both zero, it must vanish everywhere.

7. Griffiths 4.38. We'd like to relate the susceptibility (which is the response of the material to the total macroscopic electric field $\vec{P} = \epsilon_0\chi_e\vec{E}$) to the atomic polarizability (which is defined by the response of an atom to only the part of the macroscopic electric field not due to the atom itself, $\vec{p} = \alpha\vec{E}_{\text{else}} = \alpha(\vec{E} - \vec{E}_{\text{atom}})$). Thus, we need to find the contribution of the atom itself to the total macroscopic electric field. Suppose that each atom occupies a sphere of radius R, so in particular, $N = \frac{1}{4\pi R^3/3}$. Now, we will the result of problem 3.41 of Griffiths (which we will prove later) that the average field inside a sphere of radius R due to all the charge within the sphere is given by,

$$\vec{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{R^3}, \tag{43}$$

where $\vec{p}$ is the dipole moment of all the charge in the sphere (i.e., the dipole moment of the atom). As we assume that the macroscopic field does not vary significantly over a single atomic radius, we can safely assume that the contribution of the atom to the macroscopic field is just this average field,

$$\vec{E}_{\text{atom}} = \vec{E}_{\text{ave}} = -\frac{N\vec{p}}{3\epsilon_0}. \tag{44}$$

Thus, we have,

$$\vec{E} = \vec{E}_{\text{else}} + \vec{E}_{\text{atom}} = \vec{E}_{\text{else}} - \frac{N\vec{p}}{3\epsilon_0} = \vec{E}_{\text{else}} - \frac{N\alpha}{3\epsilon_0}\vec{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\vec{E}_{\text{else}}. \tag{45}$$

So, we find

$$\vec{P} = N\vec{p} = N\alpha\vec{E}_{\text{else}} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}}\vec{E} = \epsilon_0\chi_e\vec{E}, \tag{46}$$

which means that,

$$\chi_e = \frac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}. \tag{47}$$

Now, let use prove equation (43). Consider a single charge at some point $\vec{r}$ within the sphere. By Coulomb's Law, its electric field is just,

$$\vec{E}(\vec{r}') = \frac{q(\hat{\mathbf{r}}' - \hat{\mathbf{r}})}{4\pi\epsilon_0|\vec{r}' - \vec{r}|^2} \tag{48}$$

The average electric field inside the sphere B_R produced by this charge is just,

$$\vec{E}_{\text{ave}} = \frac{1}{4\pi R^3/3}\int_{B_R}\vec{E}(\vec{r}')d\tau' = \frac{1}{4\pi R^3/3}\int_{B_R}\frac{q(\hat{\mathbf{r}}' - \hat{\mathbf{r}})}{4\pi\epsilon_0|\vec{r}' - \vec{r}|^2}d\tau' = \frac{1}{4\pi\epsilon_0}\int_{B_R}\frac{-q(\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{4\pi R^3/3|\vec{r} - \vec{r}'|^2}d\tau'. \tag{49}$$

Now, note that (comparting with Griffiths 2.8) this *average* electric field is the same as the *actual* electric field at the point $\vec{r}$ produced by a uniformly charged sphere of radius R with a charge density $\rho = -\frac{q}{4\pi R^3/3}$! But because of symmetry, we can calculate this much more easily using Gauss's Law,

$$\epsilon_0\int\vec{E}\cdot da = 4\pi\epsilon_0 r^2 E_r(r) = \int\rho d\tau' = \rho 4\pi r^3/3 \Rightarrow \vec{E}_{\text{ave}}(r) = -\frac{qr\hat{\mathbf{r}}}{4\pi\epsilon_0 R^3} \tag{50}$$

Now, summing over all the charges in the atom, we see that,

$$\vec{E}_{\text{atom}} = -\frac{\sum_i q_i r_i\hat{\mathbf{r}}_i}{4\pi\epsilon_0 R^3} = -\frac{\vec{p}}{4\pi\epsilon_0 R^3}, \tag{51}$$

which is what we wanted to prove.